Understanding Conditional Probability

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Abstract

One of the most important concepts in Bayesian inference is conditional probability. In this article, we will study two interesting problems to help us to have a better understanding of conditional probability.

1 Introduction

$P(A|B)$ is called *conditional probability* which represents the probability of A is true given B is true (or simply, the probability of A given B). When A and B are independent to each other (i.e. $P(A,B) = P(A)P(B)$), we can obtain $P(A|B) = P(A)$. Conditional probability also provide a relation between joint probability and individual probabilities: $P(A,B) = P(A|B)P(B)$. Similarly we can obtain: $P(A,B) = P(B|A)P(A)$. From the above two equations, we can obtain so called Bayes' rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{1}$$

In order to have a good understanding of Bayes' rule and Bayesian learning algorithms, we will consider the following two interesting problems on conditional probability.

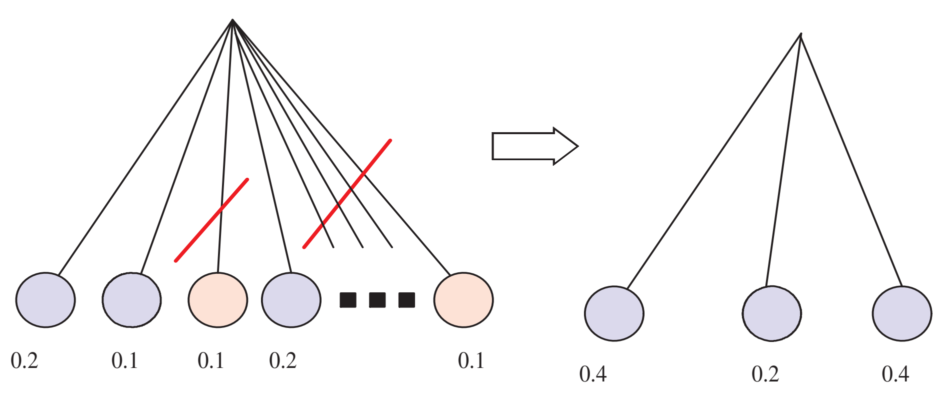

Figure 1: The tree of a probability distribution. According to the given conditions, the tree is partly cut and the distribution of the rest of leaves are re-normalized.

We can use a tree describing the possible states of the world and the possible observation, alone with the probabilities of each leaf. Condition on the event observed by setting all contradictory leaf probabilities to zero and re-normalizing the nonzero leaves. For example, see the left-hand side figure of fig.1, we cut off all the leaves which are contradictory with given conditions. The conditional probability distribution then can be represented by the tree on the right-hand side.

2 Genders of Two Children

Now let us consider the following problem about the genders of two children. This problem is taken from (Minka 2002):

Problem 1. *My neighbor has two children. It is most likely, a priori, that my neighbor has one boy and one girl, with probability* 1/2. *The other possibilities - two boys or two girls - have probabilities* 1/4 *and* 1/4 *(this is obvious!).*

(1.1) *If I ask him whether he/she has a boy, the answer is 'yes'. What is the probability of the other child is a girl?*
(1.2) *Suppose one day, I saw one of his/her children, it is a boy. What is the probability of the other child is a girl?*

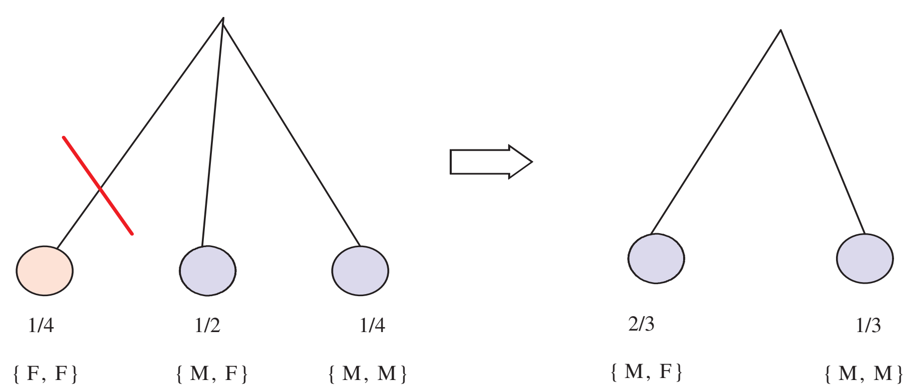

Figure 2: The tree of probability distribution on the genders of two kids.

The left-hand side figure of fig. 2 shows the tree of probability distribution on the genders of two children: {F, F}: 1/4, {M, M}: 1/4, {M, F}: 1/2. Given the condition that 'at least one of them is a boy', we delete the leaf which represents two females (i.e. {F, F}, that is contradictory to the given condition) and re-normalize the remaining leaves and we can obtain a new tree which are shown on the right-hand side of fig. 2. The probability of the other child is a girl is twice as likely for the other child is a girl than a boy.

However, instead that I happen to see one of his children is a boy. The probability that the other child is a girl is quite different from the first question. According to the rule of probabilities: observing the outcome of one coin has no affect on the other. Therefore, the probability of the other child is a girl is 1/2. The trees of probability distribution are shown in figure 3.

It seems like a paradox because it seems that in both cases we could condition on the fact that "one of his/her children is a boy". But it is not correct; we must condition on the event actually observed, not is logical implications. In the first case, the condition that one of his/her children is a boy is actually based on observation of the two children. In the second case, there is no observations at all on the second child (Minka 2002). For the first problem, the condition is on the two children and the probability distribution is also on the genders of two children. However, in the question 1.2, the condition is actually on one of the child-it is a boy. There is no more information or conditions on the second child, that is why these two problems are not the same. Question 1.2 is equivalent to

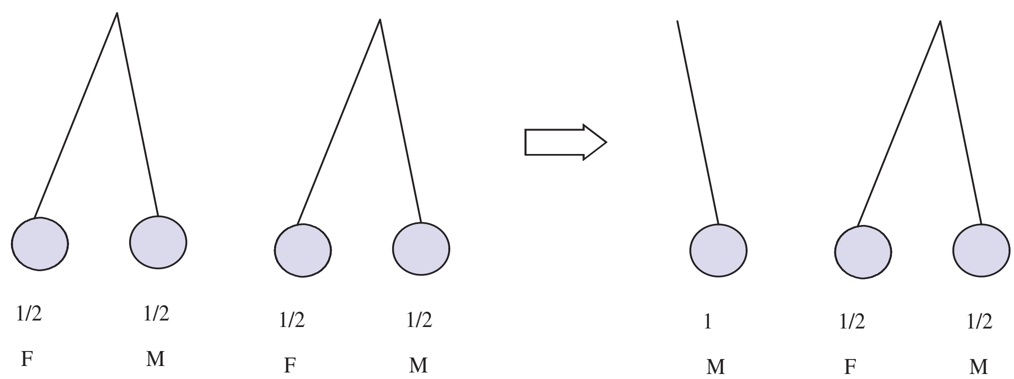

Figure 3: The trees of probability distributions on the genders of one kid and the other. Because I have seen that one child is a boy, so that one of the trees is normalized to $\{F : 0, M : 1\}$. Since there is no information about the other child, the probability distribution is unchanged.

the question: a woman hope to have two children, the first one is a boy, what is the probability of her second child is a girl? The answer is $1/2$ because the gender of the second child is not dependent on the first child.

3 Where is the Money?

Problem 2. *You are told that there is a million of dollars behind one of three doors A, B and C, and there is nothing behind the other two doors. Suppose you have chosen door A. You randomly select between door B and C and opened it. Say, door B is opened and there is nothing there. Will you stay with the same door A or change to the door C in order to get the money.*

If we denote the probability that the money is behind door i by $P(i)$ where $i \in \{A, B, C\}$. Let us first consider the following solution: the probabilities that the money is behind A, B and C are the same - i.e. $P(A) = P(B) = P(C) = 1/3$. Given a condition that door B is empty so that we delete the leaf B and re-normalize A and C. Hence, we can obtain that $P(A) = P(C) = 1/2$. Therefore, it does not matter either stay door A or change to door C because they have the same probabilities.

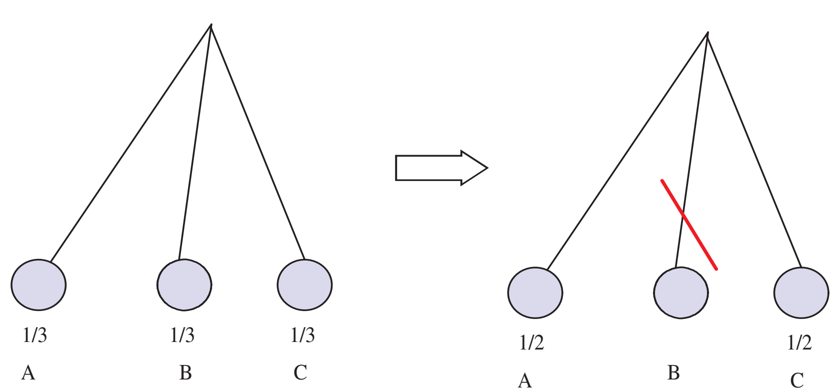

Figure 4: The tree of probability distribution after B is found to be empty. However, this is a wrong tree for the given problem.

The above solution seems to be correct by analyzing the tree of probabilities and prune based on conditions. However, it is wrong! The above solution is correct for the following question: *There is a million dollars behind one of three doors, we randomly select one door and open it. If we found it empty, what are the probabilities that the money is behind other two doors?*

For this problem, the correct answer is that we should change because the probability that the money is behind door C is twice of the probability that the money is behind door A. After door A is selected, we **randomly selected a door between** B **and** C **and the selected one was found empty**. This condition is actually on the doors B and C so that the tree of probability distribution is drawn in figure 5.

We need to notice that the probability that the money is behind a door is different from the probability that we can find the money though it is initially the same. However, after giving new evidence or conditions, the probability we can find the money is updated accroding to these conditions.

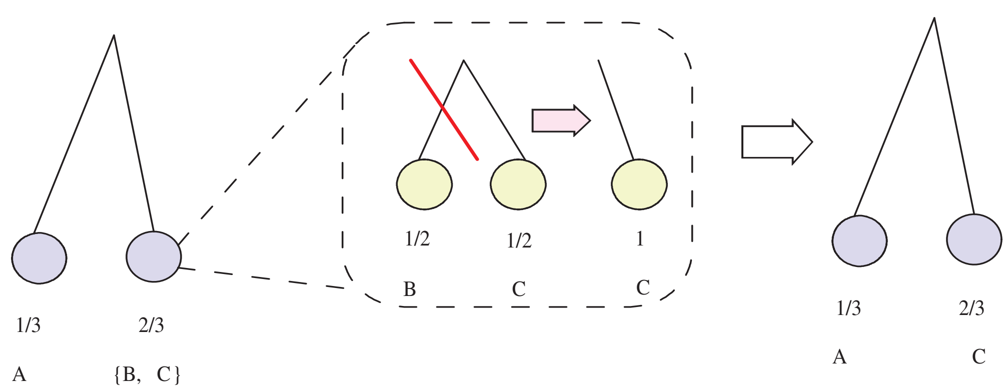

Figure 5: The tree of probability distribution. We cut off the leaf of B inside the leaf {B, C} and get a new tree on the right-hand side.

4 Interview Problem of Morgan Stanley

Problem 3. *There is one special coin with two heads in a pile of 100 coins (other 99 are normal coins with one head and one tail only). If you randomly pick one coin and see a head without looking the other side, what is the probability that this coin is the special one?*

This problem can be solved with standard Bayesian rule. Given a the random variable C (coin), it has two possible values: S (special) and N (normal). The probability of this coin is the special coin given the event seeing a head is:

$$P(C=S|H) = \frac{P(H|C=S)P(C=S)}{P(H)} \quad (2)$$

$$= \frac{1 \times (1/100)}{101/200} = \frac{2}{101} \quad (3)$$

Acknowledgement

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References

T. Minka (2002), Nuances of probability theory, Lecture Notes on Statistical Learning at CMU. *http://research.microsoft.com/~minka/statlearn/nuances.html*